

LI-YORKE SENSITIVE AND WEAK MIXING DYNAMICAL SYSTEMS

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ABSTRACT. Akin and Kolyada in 2003 [E. Akin, S. Kolyada, Li-Yorke sensitivity, *Nonlinearity* 16 (2003) 1421 – 1433] introduced the notion of Li-Yorke sensitivity. They proved that every weak mixing system (X, T) , where X is a compact metric space and T a continuous map of X is Li-Yorke sensitive. An example of Li-Yorke sensitive system without weak mixing factors was given in [M. Čiklová, Li-Yorke sensitive minimal maps, *Nonlinearity* 19 (2006) 517 – 529] (see also [M. Čiklová-Mlíčková, Li-Yorke sensitive minimal maps II, *Nonlinearity* 22 (2009) 1569 – 1573]). In their paper, Akin and Kolyada conjectured that every minimal system with a weak mixing factor, is Li-Yorke sensitive. We provide arguments supporting this conjecture though the proof seems to be difficult.

1. INTRODUCTION

A *topological dynamical system* (X, T) is a compact metric space (X, ρ) endowed with a continuous surjective map $T : X \rightarrow X$. Denote by T^n the n th iterate of T , $n \geq 0$. Points $x, y \in X$ are *proximal*, or δ -*asymptotic* (with $\delta \geq 0$), or *distal* if $\liminf_{n \rightarrow \infty} \rho(T^n(x), T^n(y)) = 0$, $\limsup_{n \rightarrow \infty} \rho(T^n(x), T^n(y)) \leq \delta$, or $\liminf_{n \rightarrow \infty} \rho(T^n(x), T^n(y)) > 0$, respectively; instead of 0-asymptotic we say *asymptotic*. A map $T : X \rightarrow X$ is *Li-Yorke sensitive*, briefly *LYS* or *LYS $_\varepsilon$* , if there is an $\varepsilon > 0$ with the property that every $x \in X$ is a limit of points $y \in X$ such that the pair (x, y) is *proximal* but not ε -*asymptotic*, i.e., if

$$(1) \quad \liminf_{n \rightarrow \infty} \rho(T^n(x), T^n(y)) = 0, \quad \text{and} \quad \limsup_{n \rightarrow \infty} \rho(T^n(x), T^n(y)) > \varepsilon.$$

Every pair $(x, y) \in X \times X$ satisfying (1) is an ε -*Li-Yorke pair*. A set $S \subseteq X$ such that any points $x \neq y$ in S satisfy (1) is an (ε) -*scrambled set*. A map T is *Li-Yorke chaotic*, briefly *LYC* or *LYC $_\varepsilon$* if it has an uncountable ε -scrambled set, for some $\varepsilon > 0$.

A system (X, T) is *transitive* if for every pair of open, nonempty subsets $U, V \subset X$ there is a positive integer n such that $U \cap T^{-n}(V) \neq \emptyset$;

Date: September 14, 2016.

it is *weakly mixing* if the product system $(X \times X, T \times T)$ is transitive; it is *minimal* if every point $x \in X$ has a dense orbit $\{T^n(x)\}_{n=0}^\infty$. Finally, a system (Y, S) is a *factor* of (X, T) if there is a surjective continuous map $\pi : X \rightarrow Y$ such that $\pi \circ T = S \circ \pi$. In this case we say that (X, T) is an *extension* of (Y, S) . A *skew-product system* is a system $(X \times Y, F)$ where X, Y are compact metric spaces, and F a continuous map such that $F(x, y) = (f(x), g_x(y))$, for every $x \in X, y \in Y$. Other notions will be defined later, or can be found in [1] or in related papers listed in references.

The notion of *LYS* was introduced and studied by Akin and Kolyada [2]. It turns out that such systems are related to weak mixing systems. For example, every nontrivial weak mixing system is *LYS*. Therefore, Akin and Kolyada stated in [2] five conjectures concerning *LYS* systems. Three of them were disproved in [5] and [6]. In particular, it was proved that a minimal *LYS* system need not have a nontrivial weak mixing factor, and that a minimal system with a nontrivial *LYS* factor need not be *LYS*. The remaining two open problems are the following:

- P1.** Is every minimal system with a nontrivial weak mixing factor *LYS*?
P2. Does *LYS* imply *LYC*?

Both problems seem to be difficult but, in contrast to the preceding ones, it seems that the answer is in both cases positive. In this paper we give partial solutions. Recall that the only known result related to (P1) is the following

Theorem 1. (See [2], [3].) *If (X, T) is minimal weak mixing and (Y, S) minimal and distal then $(X \times Y, T \times S)$ is minimal and *LYS*.*

We generalize it to some skew-product extensions of the original system in Theorems 10, 11 which represent the main results of this paper. Then, in Theorem 18 we show that the restriction to skew-product extensions is not too limiting. Finally, our last result, Theorem 19, essentially diminishes the possible class of systems which may not satisfy (P2). For convenience, we recall several known results which will be of use in the next sections.

Lemma 2. (See [9].) *If (X, T) is a minimal system then, for every open set $G \subseteq X$ there is an open set $H \subseteq X$ such that $H \subseteq T(G) \subseteq \overline{H}$.*

Lemma 3. (See [2].) *If (X, T) is weak mixing then, for every $x \in X$, the set of points $y \in X$ which are proximal but not ε -asymptotic to x , is a dense G_δ subset of X .*

Lemma 4. (See [8].) *Let X be a complete separable metric space without isolated points. If $R \subseteq X \times X$ is a symmetric relation with the property that for each $x \in X$, $R(x) = \{y \in X; (x, y) \in R\}$ contains a dense G_δ subset, then there is a dense uncountable set $D \subseteq X$ such that $D \times D \setminus \Delta \subset R$, where Δ is the set of pairs (x, x) , $x \in X$.*

The following is a topological version of the Fubini Theorem.

Lemma 5. (See [1] or [8].) *Let R be a relation on a complete separable metric space X which contains a dense G_δ subset of $X \times X$. Then there is a dense G_δ set $A \subseteq X$ such that for each $x \in A$, there exists a dense G_δ set $X_x \subseteq X$ with $\{(x, y); x \in A, y \in X_x\} \subseteq R$.*

Lemma 6. (See [8] and [2].) *If (X, T) is LYS then, for some $\delta > 0$, the set of δ -asymptotic pairs is a first category subset of $X \times X$.*

2. MINIMAL FINITE-TYPE SKEW-PRODUCT EXTENSIONS OF WEAK MIXING SYSTEMS

Lemma 7. *Let (X, T) be minimal, $A = \{a_1, \dots, a_m\}$ a finite set with discrete topology, $Y = X \times A$, S a skew-product map $Y \rightarrow Y$ so that $S(u, v) = (T(u), G_u(v))$, and $M \subseteq Y$ a minimal set. Then*

- (1) *for every $u \in X$, the map G_u restricted to the set $M_u := M \cap (\{u\} \times A)$, is injective;*
- (2) *there is a nonempty set $B \subseteq A$ such that $(M, S|_M)$ is conjugate to $(X \times B, S|_{X \times B})$.*

Proof. (i) Assume $G_u(v_1) = G_u(v_2) = v$, for some u in X and $v_1 \neq v_2$ in A . Since A is discrete, by the continuity of S there is a neighborhood U of u such that, for every $w \in U$, $G_w(v_1) = G_w(v_2) = v$. Then $U_1 = U \times \{v_1\}$ and $U_2 = U \times \{v_2\}$ would be disjoint nonempty open sets with $S(U_1) = S(U_2)$. But this is impossible, by Lemma 2.

(ii) Let $\mathcal{H} = \{h_1, \dots, h_l\}$ be the collection of maps $G_u|_{M_u}$, $u \in X$. By the continuity, there is a decomposition of X into clopen sets X_1, \dots, X_l such that, for every $u \in X_j$, $G_u|_{M_u} = h_j$. Let c_i be the number of points in the domain of h_i and let, say, $c_1 \geq c_i$, for every i . Since T is transitive, (i) implies $c_i = c_1$, for every i . It follows that $M = X_1 \times A_1 \cup X_2 \times A_2 \cup \dots \cup X_l \times A_l$, where $\#A_j = c_1$, for every j . Hence M is conjugate to $X \times A_1$. \square

Lemma 8. *If (X, T) is minimal weak mixing, then for every $x \in X$ the set $\text{Tran}(x) \subset X$ of points y such that (x, y) is a transitive point with respect to $T \times T$, is a dense G_δ set.*

Proof. Let $x_0 \in X$ be given. The set $\text{Tran}(x_0)$ of points $y \in X$ such that (x_0, y) is a transitive point of $T \times T$, is a G_δ set since it is the intersection of two G_δ sets, the set of transitive points $(x, y) \in X \times X$, and $\{x_0\} \times X$. So it suffices to show that $\text{Tran}(x_0)$ is dense in X . Let $\{G_n\}_{n \geq 1}$ be a base of open sets for $X \times X$ of the form $G_n = I_n \times J_n$, where I_n, J_n are open sets. Let $U_0 \subset X$ be nonempty open. By induction, there are nonempty open sets $U_0 \supset U_1 \supset U_2 \supset \dots$, and a sequence $n_1 < n_2 < \dots$ of positive integers such that

$$(2) \quad \overline{U}_j \subset U_{j-1}, \quad T^{n_j}(x_0) \in I_j \text{ and } T^{n_j}(U_j) \subset J_j, \quad j \in \mathbb{N}.$$

Indeed, since T is minimal, there is a $k_1 > 0$ such that, for every j , there is an s , $0 \leq s < k_1$, with $T^{j+s}(x_0) \in I_1$. Since T is weak mixing, the set $N(U_0, J_1)$ of times i such that $T^i(U_0) \cap J_1 \neq \emptyset$, contains arbitrarily long blocks of successive integers. It follows that there is an n_1 such that $T^{n_1}(x_0) \in I_1$ and $T^{n_1}(U_0) \cap J_1 \neq \emptyset$. Since T is minimal and U_0 open, $T^{n_1}(U_0)$ has nonempty interior (see Lemma 2) and hence $T^{n_1}(U_0) \cap J_1$ contains a nonempty open set H . It suffices to take for U_1 a nonempty open set such that $\overline{U}_1 \subset T^{-n_1}(H)$. Thus, we have n_1 and U_1 satisfying (2) (for $j = 1$). Next we apply the above process with U_0 replaced by U_1 , G_1 by G_2 , obtaining $U_2 \subset U_1$ and $n_2 > n_1$, etc. This proves (2).

To finish the argument put $Y = \bigcap_{j \geq 1} U_j = \bigcap_{j \geq 1} \overline{U}_j$. Then $Y \neq \emptyset$ is a G_δ set and, by (2), for every $y \in Y$, (x_0, y) is a transitive point. \square

For $\xi > 0$ let $\Delta_\xi \subseteq X \times X$ be the set of pairs (x, y) such that $\rho(x, y) < \xi$.

Theorem 9. *Let (X, T) be a minimal weak mixing topological dynamical system. Let A be a finite space with discrete topology, $Y = X \times A$ with the max-metric, and (Y, S) a skew-product extension of (X, T) such that $S(t, a) = (T(t), G_t(a))$, where every fibre map G_t is a bijection of A . Then (Y, S) is LYS_ε for any $0 < \varepsilon < \text{diam}(X)$.*

Proof. The following terminology and notation will be useful. For every $z = (x, y) \in X \times X$ and $i \in \mathbb{N}$, denote by $(T \times T)^i(z) = (x_i, y_i)$ the i th iterate of z , with $x_0 := x$, $y_0 := y$. Let $g_0 = h_0 = \text{Id}$, the identity and, for $i > 0$ let $g_i = G_{x_{i-1}} \circ G_{x_{i-2}} \circ \dots \circ G_{x_0}$, similarly let h_i be the composition of $i - 1$ corresponding maps G_{y_j} , and let $c_i := (g_i, h_i)$. For $\eta > 0$ let $N = N(x, y, \eta) = \{i \in \mathbb{N}; (x_i, y_i) \in \Delta_\eta\}$. The sequence $\{c_i\}_{i \in N}$ is the η -characteristic sequence of (x, y) . Let $j_0 < j_1 < \dots$ be the numbers in N . A finite string $c_{j_0}, c_{j_1}, \dots, c_{j_{k-1}}$ is an η -saturated chain for (x, y) of length k if the string contains all members of the η -characteristic sequence of (x, y) ; we denote it as

$M(x, y, \eta)$, and we let $C(x, y, \eta)$ denote the set of elements in $M(x, y, \eta)$. Notice that we do not determine uniquely the length of a saturated string: if $M(x, y, \eta) = \{c_{j_0}, \dots, c_{j_{k-1}}\}$ then $\{c_{j_0}, \dots, c_{j_k}, c_{j_k}\}$ is also saturated string. When dealing with an another pair, (x', y') , we use primes to distinguish the related symbols like $x'_i, y'_i, j'_i, c'_{j'_i}, k'$, etc. By the continuity, for every saturated chain $M(x, y, \eta)$ of length k there is an open neighborhood $U(x, y, \eta)$ of (x, y) such that, for any pair $(x', y') \in U(x, y, \eta)$, $\{(x'_i, y'_i)\}_{0 \leq i \leq j_{k-1}}$ traces $\{(x_i, y_i)\}_{0 \leq i \leq j_{k-1}}$ so that $\rho((x_i, y_i), (x'_i, y'_i)) < \eta$ and $(x'_i, y'_i) \in \Delta_\eta$ for $i \in \{j_0, j_1, j_2, \dots, j_{k-1}\}$. In particular, $M(x, y, \eta) = M(x', y', \eta)$.

Saturated strings $M(x, y, \eta), M(x', y', \eta')$ with $\eta' \leq \eta$ of two *transitive* pairs (x, y) and (x', y') of length k and k' , respectively, can be joined in a single chain of (x, y) in the following sense. Since (x, y) is transitive, there is an $n \geq j_{k-1}$ such that $(x_n, y_n) \in U(x', y', \eta')$. It follows that $n = j_s$ for some $s \geq k-1$, and the trajectory $\{(x_{n+i}, y_{n+i})\}$ traces the trajectory $\{(x'_i, y'_i)\}$ for $i \in [0, j'_{k'-1}]$, remaining within distance $\eta' \leq \eta$ for $i = j'_l, 0 \leq l \leq k'-1$. We denote the resulting string as $M(x, y, \eta) * M(x', y', \eta')$. Its length is $s+k'$. Thus, we have the following

Claim 1. *Let $\eta \geq \eta' > 0$, and $M(x, y, \eta), M(x', y', \eta')$ be saturated strings of transitive pairs, of length k and k' , respectively. Then*

(i) *The string $M(x, y, \eta) * M(x', y', \eta')$ need not be saturated for η or η' , but it is obtained from $M(x, y, \eta)$ of sufficiently high length by omitting some elements;*

(ii) $C(x, y, \eta) \supseteq C(x', y', \eta') \circ c_{j_s}$,

where j_s is specified above (c_{j_s} is the element “connecting” both saturated strings), and $\{f, g\} \circ h$ means $\{f \circ h, g \circ h\}$.

Claim 2. *Let $(x, y), (x', y')$ be transitive pairs. Then*

(i) $\#C(x, y, \eta) = \#C(x', y', \eta)$ for any $\eta > 0$;

(ii) *there is a $\xi > 0$ such that if $(x, y), (x', y') \in \Delta_\xi$ and $0 < \eta, \eta' < \xi$ then $\#C(x, y, \eta) = \#C(x', y', \eta')$.*

Proof of Claim 2. (i) Assume $m := \#C(x, y, \eta) < m' := \#C(x', y', \eta)$. Then $\#C(x', y', \eta) = m'$ and, by Lemma B(ii), $C(x, y, \eta)$ contains m' distinct elements, a contradiction.

(ii) We may assume $\eta' < \eta$. Then obviously $\#C(x, y, \eta') \leq \#C(x, y, \eta)$. Since G is finite, there is a $\xi > 0$, and $m_0 \geq 1$ such that $\#C(x, y, \eta) = m_0$ whenever $\eta < \xi$. To finish apply (i). \square

Claim 3. *Let ξ be as in Claim 2, and $(x, y) \in \Delta_\xi$ be a transitive pair, and $\eta' < \eta := \xi$. If $c_{j_s} \in C(x, y, \eta)$ is the element connecting the strings $M(x, y, \eta)$ and $M(x, y, \eta')$, then $C(x, y, \eta') \circ c_{j_s}$ contains the identity map (Id, Id) .*

Proof of Claim 3. Let x, y be as in the hypothesis. By Claim 1, $M(x, y, \eta) * M(x, y, \eta')$ contains the string $M(x, y, \eta') \circ c_{j_s}$. But $M(x, y, \eta') \circ c_{j_s}$ must contain the identity. To see this note that, by definition, the first member c_0 of the η -characteristic sequence is the identity. Since $\rho(x, y) < \eta (= \xi)$, c_0 is also the first member of $M(x, y, \eta)$, i.e., $j_0 = 0$. By Claim 2 (ii), $C(x, y, \eta)$ has the same cardinality as $C(x, y, \eta')$ hence as $C(x, y, \eta') \circ c_{j_s}$, since c_{j_s} is a bijection. Thus $C(x, y, \eta') \circ c_{j_s}$ contains the identity map (Id, Id) . \square

Finally, let $x \in X$ and U be an arbitrary neighborhood of x . Assume that $\xi > 0$ is as in Claim 2. By Lemma 8 there is a point $y \in U$ such that $(x, y) \in \Delta_\xi$ is transitive with respect to $T \times T$. Since Y is equipped with the max-metric it suffices to prove that if $a \in A$ then, for every $0 < \eta' < \xi$, $w = ((x, a)(y, a))$ is an η' -proximal pair. This follows by Claim 3. $\square\square$

Theorem 10. *Let (X, T) be minimal weak mixing, and $A \neq \emptyset$ a finite metric space. Let S be a skew-product map of $X \times A$. Then, for every minimal set $M \subseteq X \times A$ of S , $(M, S|_M)$ is LYS_ε , for some $\varepsilon > 0$. Moreover, S is LYC_ε and for any $a \in A$, there is a dense ε -scrambled set $D_a \subset X \times \{a\}$ of type G_δ .*

Proof. The first part follows by Lemmas 3, 7, 8 and Theorem 9, the last statement by Lemma 4. To finish the argument, fix an $a \in A$, and let $Y := X \times \{a\}$. Denote by R the set of ε -Li-Yorke pairs (x, y) in $Y \times Y$. By Lemma 4 there is an uncountable dense scrambled set $D_a \subset Y$ such that every distinct points in D_a form an ε -Li-Yorke pair. \square

3. INFINITE TYPE SKEW-PRODUCT EXTENSIONS OF WEAK MIXING SYSTEMS

Theorem 9 and hence, Theorem 10 can be generalized to certain types of skew-product maps of $X \times A$, where (X, T) is minimal weak mixing, and A is infinite compact. As a sample we provide the following. Recall that an *adding machine* or *odometer* related to a sequence p_1, p_2, \dots of primes is a system (X, τ) , where $X = \prod_{j \geq 1} X_j$, $X_j = \{0, 1, \dots, p_j - 1\}$, and $\tau(x_1 x_2 x_3 \dots) = x_1 x_2 x_3 \dots + 1000 \dots$,

when adding is modulo p_j at the j th position from the left to the right, see, e.g., [4]. Obviously X is a Cantor-type set.

Theorem 11. *Let (X, T) be minimal weak mixing, Y a Cantor-type set, and $S : X \times Y \rightarrow X \times Y$ a skew-product map, $S(x, y) = (T(x), R_x(y))$ such that, for every $x \in X$, R_x is an odometer, or the identity. Then $(X \times Y, S)$ is LYS.*

Proof. It is similar to the proof of Theorem 10. It is based on the following Lemma 12, and on Theorem 9 and Lemma 8. \square

Lemma 12. *Let X, Y and S be as in Theorem 11. Then for every $\delta > 0$ there is an $m > 0$ such that, for every $x \in X$, $y \in Y$ and $k \in \mathbb{N}$, $|y - R_x^{km}(y)| < \delta$.*

Proof. It suffices to show that, for every $\delta > 0$ and for every $x \in X$, there is a decomposition of Y into clopen portions Y_1, Y_2, \dots, Y_m forming an R_x -periodic orbit, such that the diameter of every Y_j is less than δ . Assume the contrary. Then there is an increasing sequence $m_1 < m_2 < \dots$ of positive integers, a sequence x_1, x_2, \dots in X , and a sequence Y_1, Y_2, \dots of clopen portions of Y such that, for every j , $\text{diam}(Y_j) \geq \delta$ and Y_j is a periodic portion with respect to R_{x_j} of period m_j . Taking a subsequence if necessary, we may assume that $\lim_{j \rightarrow \infty} x_j = x_0$, and $\lim_{j \rightarrow \infty} Y_j = Y_0$. Then Y_0 is a compact portion of Y with “infinite” period, i.e., $R_{x_0}^n(Y_0)$ is disjoint from Y_0 , for every $n > 0$, contrary to the assumption that R_{x_0} is an odometer, or the identity. \square

4. FINITE-TYPE EXTENSIONS AND SKEW PRODUCT SYSTEMS

Here we show that the assumption in Theorem 10, that the corresponding map is a skew-product map on $Y \times \{1, 2, \dots, n\}$ is not too restrictive since, for certain but not all types of minimal weak mixing systems every n to one extension is a skew-product map, see Theorem 18. On the other hand, by the next lemma and the subsequent remark, not every finite type extension of a minimal weak mixing system is (conjugate to) a skew-product map. Recall (see, e.g., [10] for details) that a *continuum* is a nonempty connected compact metric space. A continuum X is *unicoherent* if for every two continua A, B with $A \cup B = X$ the set $A \cap B$ is connected.

Lemma 13. *Let (Y_1, ρ_1) be a continuum which is not unicoherent. (In particular, let Y_1 be the circle). Let $Y = Y_0 \times Y_1$, where (Y_0, ρ_0) is a compact metric space. Finally, let S be a continuous map $Y \rightarrow Y$. Then for every $k \in \mathbb{N}$ there is an isometric extension (X, T) of (Y, S) ,*

with factor map $\pi : X \rightarrow Y$ such that $\#\pi^{-1}(y) = k$ for every $y \in Y$, and (X, T) is not conjugate to a skew-product map $Y \times \{1, \dots, k\} \rightarrow Y \times \{1, \dots, k\}$.

Proof. Since Y_1 is not unicoherent there are continua $A, B \neq Y_1$ such that $A \cap B$ is not connected. Hence, $A \cap B = C \cup D$ where C, D are nonempty disjoint compact sets. Assume first that $A \cap B = \{a, b\}$. Let $k > 1$ be an integer, $K_0 \subset \mathbb{S}$ the set of points on the unit circle representing the k th roots of 1 and, for any $t \in I$, let φ_t be the rotation of the set \mathbb{S} at angle $2t\pi/k$ in the positive direction. Thus, $\varphi_1(K_0) = K_0$. For $v \in B$ let $t(v) = \rho_1(a, v)/\rho_1(a, b)$. Let $X_1 \subset Y_1 \times \mathbb{S}$ be given by $X_1 = (A \times K_0) \cup \bigcup_{v \in B \setminus A} (\{v\} \times \varphi_{t(v)}(K_0))$, and let $X = Y_0 \times X_1$; obviously, X_1 is connected. For a $y \in Y$ denote by y' the projection of y onto Y_1 and let $T : X \rightarrow X$ be such that, for $(y, z) \in X$ with $y \in Y$,

$$(3) \quad T(y, z) = \begin{cases} (S(y), z) & \text{if } y', S(y)' \in A, \\ (S(y), \varphi_{t(S(y)')-t(a)}(z)) & \text{if } y' \in A, S(y)' \in B \setminus A, \\ (S(y), \varphi_{t(b)-t(y')}(z)) & \text{if } y' \in B \setminus A, S(y)' \in A, \\ (S(y), \varphi_{t(S(y)')-t(y')}(z)) & \text{if } y', S(y)' \in B \setminus A. \end{cases}$$

Then T is a continuous bijection $X \rightarrow X$. To finish the argument assume (X, T) is conjugate to $Y \times \{1, \dots, k\}$. But then X_1 should be homeomorphic to $Y_1 \times \{1, \dots, k\}$, which is impossible since the first space is connected while the second one has k disjoint connected components.

In the general case when C or D is not a singleton, the argument is similar, we only take $t(v) = \text{dist}(C, v)/\text{dist}(C, D)$ for $v \in Y_1 \setminus A$. \square

Remark 14. In [7] there is an example of a minimal weak mixing (Y, S) , where Y is the 5th-dimensional torus. By the previous lemma, even in this case, there is a minimal k -extension (X, T) of (Y, S) which is not conjugate to a skew-product system.

Lemma 15. Let (X, ρ) be a compact metric space such that every connected component of X is nowhere dense in X . Then, for every $\delta > 0$ there is a finite decomposition $X_1 \cup X_2 \cup \dots \cup X_m$ of X into disjoint compact subsets such that, for every j , X_j is a subset of the δ -neighborhood of a connected component of X .

Proof. Let X_δ be the union of connected components of X with diameter $\geq \delta$. Then X_δ is a closed set. Indeed, let $x_n \in X_\delta$ be such that $\lim_{n \rightarrow \infty} x_n = x$. Then there are connected components $K_n \subseteq X_\delta$ such that $x_n \in K_n$, for every n . Since the set of nonempty compact subsets of X , with the Hausdorff metric ρ_H , is a compact set, we may assume

that there is a compact set $K \subset X$ such that $\lim_{n \rightarrow \infty} \rho_H(K_n, K) = 0$. Since $x \in K$, it suffices to show that K is a connected component of X with $K \subset X_\delta$. Obviously, $\text{diam}(K) \geq \delta$. To show that K is connected, assume the contrary. Then there are disjoint closed sets G, H with $K = G \cup H$ such that $K \cap G \neq \emptyset \neq H \cap K$. Let G', H' be disjoint closed neighborhoods of G and H , respectively. Then, for every sufficiently large n , $K_n \subseteq G' \cup H'$, $K_n \cap G' \neq \emptyset \neq K_n \cap H'$ which is a contradiction. Thus, X_δ is a closed set.

Next we show that for every connected component $K \subset X_\delta$ there is a compact neighborhood $U(K) = U$ of K such that $X \setminus U$ is compact, and U is contained in the open δ -neighborhood V of K . Since K is a component, for every $x \in X \setminus V$ there is a decomposition of X into disjoint compact sets G_x, H_x such that G_x is a neighborhood of x and $K \subseteq H_x$. Since $X \setminus V$ is compact, there is a finite cover $G_{x_1} \cup G_{x_2} \cup \dots \cup G_{x_k} = X \setminus V$. Take $U(K) = H_{x_1} \cap \dots \cap H_{x_k}$.

To finish the proof it suffices to take a finite cover $W = W_1 \cup W_2 \cup \dots \cup W_s$ of X_δ consisting of disjoint compact sets such that every W_j is a set $U(K_j)$ with $K_j \subseteq X_\delta$ a connected component. Then $X \setminus W$ is a compact set which can be divided into finitely disjoint compact sets with $\text{diam} \leq \delta$. \square

We say that a set A in a metric space is δ -separated if $\rho(u, v) \geq \delta$ for every distinct $u, v \in A$.

Lemma 16. *Let X, Y be compact metric spaces, $n > 0$ an integer, and $\pi : X \rightarrow Y$ a continuous map such that, for every $y \in Y$, $\#\pi^{-1}(y) = n$. Assume that (Y, S) is a minimal system. Then there is a $\delta_0 > 0$ such that every set $\pi^{-1}(y)$ is δ_0 -separated.*

Proof. For $\delta > 0$ let Y_δ be the set of $y \in Y$ such that $\pi^{-1}(y)$ is δ -separated. Then Y_δ is a compact set. Indeed, let $y_j \in Y_\delta$ such that $\lim_{j \rightarrow \infty} y_j = y_0$. Since the space of nonempty compact subsets of X , equipped with the Hausdorff metric ρ_H , is a compact space there is a subsequence $j_1 < j_2 < \dots$ and a set $A \subset X$ such that $\lim_{k \rightarrow \infty} \rho_H(\pi^{-1}(y_{j_k}), A) = 0$. By the continuity, $\pi(A) = y_0$ and $\#A = n$. Hence $A = \pi^{-1}(y_0)$ is δ -separated, i.e., $y_0 \in Y_\delta$. Since every $\pi^{-1}(y)$ is finite, $\bigcup_{j > 0} Y_{1/j} = Y$. By the Baire category theorem there is a $k > 0$ such that $Y_{1/k}$ has nonempty interior. Since Y is minimal, there is an $m > 0$ such that $\bigcup_{0 \leq j \leq m} S^{-j}(Y_{1/k}) = Y$. By the continuity of S there is a $\delta_0 > 0$ such that $S^{-j}(Y_{1/k}) \subset Y_{\delta_0}$, $0 \leq j \leq m$. Consequently, $Y = Y_{\delta_0}$. \square

Lemma 17. *Let (Y, S) be a factor of (X, T) , with factor map $\pi : X \rightarrow Y$. Assume that (Y, S) is minimal, weak mixing, not connected,*

and such that every subcontinuum of Y is unicoherent. Finally, let $n > 0$ be an integer such that, for every $y \in Y$, $\#\pi^{-1}(y) = n$. Then (X, T) is conjugate to a skew-product map $F : Y \times N \rightarrow Y \times N$, where $N = \{1, 2, \dots, n\}$.

Proof. By Lemma 16 there is a $\delta_0 > 0$ such that $\pi^{-1}(y)$ is δ_0 -separated, for every $y \in Y$. Let $0 < \eta < \delta_0/3$ be such that, for every $u, v \in X$, $\rho(u, v) < 2\eta$ implies $\rho(T(u), T(v)) < \delta_0/3$. Since (Y, S) is weak mixing and not connected every connected component of (Y, ρ) is nowhere dense. By Lemma 15, there is a finite decomposition $Y_1 \cup Y_2 \cup \dots \cup Y_m$ of Y into disjoint compact sets such that Y_j is contained in the η -neighborhood of a connected component P_k of Y . Let $p_k \in P_k$ and let U_0 be a compact η -neighborhood of p_k . Define a continuous map $\psi_k : \pi^{-1}(U_0 \cap P_k) \rightarrow (U_0 \cap P_k) \times N$. If $Y_k \subset U_0$ extend ψ_k continuously onto $\pi^{-1}(Y_k)$; by the choice of η this extension is uniquely determined by ψ_k restricted to $\pi^{-1}(U_0 \cap P_k)$. Otherwise take U_1 the compact η -neighborhood of U_0 and extend ψ_k continuously to a map $\pi^{-1}(U_1 \cap P_k) \rightarrow (U_1 \cap P_k) \times N$, etc. Since P_k is compact after finite number of steps ψ_k is continuously extended onto $\pi^{-1}(P_k)$ such that $\psi_k(\pi^{-1}(P_k)) = P_k \times N$. Since P_k is unicoherent continuum, this extension is uniquely determined by ψ_k on $\pi^{-1}(U_0 \cap P_k)$. Finally, by the choice of η , ψ_k can be continuously (and uniquely) extended onto $\pi^{-1}(Y_k)$. To finish the argument take $\psi = \psi_1 \cup \dots \cup \psi_m$ which is a continuous bijective map $X \rightarrow Y \times N$, and take $F = \psi \circ T \circ \psi^{-1}$. \square

Theorem 18. *Let (Y, S) be minimal, weak mixing, not connected, and such that every subcontinuum of Y is unicoherent. Let $n > 0$ be an integer, and let (X, T) be an extension of (Y, S) such that $\#\pi^{-1}(y) = n$ for every $y \in Y$. Finally, let $M \subseteq X$ be a minimal set. Then $(M, T|_M)$ is LYS.*

Proof. It follows by Lemma 17 and Theorem 10. \square

5. LI-YORKE SENSITIVITY AND LI-YORKE CHAOS

In [2] there is a problem whether *LYS* implies *LYC*; the converse implication obviously is not true. Here we show that under some additional conditions, the answer is positive. This significantly restricts the class of systems (X, T) for which the implication need not hold. To simplify the argument, we will use the following notation. Given a system (X, T) denote by *Dist* the set of distal pairs $(x, y) \in X \times X$, and by *Asym $_\varepsilon$* the set of ε -asymptotic pairs (x, y) in $X \times X$.

Theorem 19. *Let (X, T) be LYS. Assume there is a non-empty open set $H \subset X$ such that $(H \times H) \cap \text{Dist}$ has empty interior or (equivalently) that $(H \times H) \cap \text{Dist}$ is a set of the first Baire category. Then there is an $\varepsilon > 0$ such that (X, T) is LYC_ε .*

Proof. By Lemma 6 there is an $\varepsilon > 0$ such that T is LYS_ε , and Asym_ε is a first category set. It is easy to see that Dist and Asym_ε are F_σ sets hence, by the Baire category theorem, $(H \times H) \cap \text{Dist}$ is of the first category if and only if it has the empty interior. Assume $(H \times H) \cap \text{Dist}$ is a first category set and put $L = X \times X \setminus (\text{Dist} \cup \text{Asym}_\varepsilon)$. Then L is a G_δ set dense in $H \times H$. By Lemmas 5 and 4, there is an uncountable set $D \subset H$ such that $D \times D \setminus \Delta \subset L$. Obviously, L is the set of ε -Li-Yorke pairs in $X \times X$. Hence, D is an ε -scrambled set for (X, T) and hence, T is LYC_ε . \square

Remark 20. *For a minimal (X, T) which is both LYS and LYC, the set Dist can be very large. In [5] there is an example of such a system, even without a weak mixing factor such that the set Dist contains an open dense subset of $X \times X$.*

ACKNOWLEDGEMENTS

The author would like to express her thanks to Zuzana Roth and Samuel Roth for their essential conception. They proved Theorem 9 for a set A containing exactly two points, which contributed to prove of Theorem 9. The author also sincerely thanks Professor Jaroslav Smítal for his very kind suggestions and excellent support.

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